Operations of Matrices

- A matrix having m rows and n columns has the dimensions m x n.
- A matrix is square if m = n.

$$\begin{bmatrix} 0 & 2 & 4 \\ -1 & 3 & 0 \end{bmatrix}$$
2×3

Ex.1 What are the dimensions of the matrix?

a) $\begin{bmatrix} 1 & -3 & 0 & \frac{1}{2} \end{bmatrix}$ 1×4

b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ 2×2

c) $\begin{bmatrix} 5 & 0 \\ 2 & -2 \\ -7 & 4 \end{bmatrix}$ 3×2

Addition of Matrices

- You can add Matrices (of the same dimensions) by adding their corresponding parts.

Ex.2 $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$

Ex.3 $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$

Scalar Multiplication and Matrix Subtraction

- You can multiply a matrix by a scalar c by multiplying each entry by c.

Ex.4 $3\begin{bmatrix} 2 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}$

Ex.5 $\begin{bmatrix} 6 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix} - \begin{bmatrix} -2 & 0 & 0 \\ -1 & +4 & -3 \\ +1 & -3 & -2 \end{bmatrix}$

$$\begin{bmatrix} 4 & 6 & 12 \\ -10 & 4 & -6 \\ 7 & 0 & 4 \end{bmatrix}$$

Distributive Property

Ex.6 Evaluate the expression

$$3\left(\begin{bmatrix} -2 & 0 \\ 4 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -2 \\ 3 & 7 \end{bmatrix}\right) = 3\boxed{\begin{bmatrix} 2 & -2 \\ 7 & 8 \end{bmatrix}}$$

$$\boxed{\begin{bmatrix} 6 & -6 \\ 21 & 24 \end{bmatrix}}$$

Equality of Matrices

- Two Matrices are equal if they have the same order.
- Because two Matrices are equal only their corresponding entries are equal.

Ex.7 $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix}$

$a_{11} = 2$

$a_{12} = -1$

$a_{21} = -3$

$a_{22} = 0$

Solving Matrix Equations

Ex.8 $3X + A = B$

$A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$ $B = \begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix}$

$$3X + \cancel{A} = B$$
$$\quad\ \ -\cancel{A} \quad -A$$

$$\cancel{\tfrac{1}{3}}(\cancel{3}X) = (B-A)\tfrac{1}{3}$$

$$X = \tfrac{1}{3}(B-A)$$

$$X = \tfrac{1}{3}\left(\begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} -1 & +2 \\ -0 & -3 \end{bmatrix}\right)$$

$$= \tfrac{1}{3}\begin{bmatrix} -4 & 6 \\ 2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{4}{3} & 2 \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

Matrix Multiplication

- If the dimensions of matrix A is m X n and the dimensions of B are n X p, then the product of A X B will be m X p.
- The definition of matrix multiplication indicates a row by column Multiplication.

Ex.9 Find the product of AB

$A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix}$ $B = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$

3x2 2x2

3x2

$$\begin{bmatrix} -1(-3)+3(-4) & -1(2)+3(1) \\ 4(-3)-2(-4) & 4(2)-2(1) \\ 5(-3)+0(-4) & 5(2)+0(1) \end{bmatrix}$$

$$\begin{bmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{bmatrix}$$

Ex.10 $\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$2 \times 2 \quad 2 \times 2$

$$\begin{bmatrix} 3(1)+4(0) & 3(0)+4(1) \\ -2(1)+5(0) & -2(0)+5(1) \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}}$$

Ex.11 $\begin{bmatrix} 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$1 \times 3 \quad 3 \times 1$

$$\begin{bmatrix} 1(2)-2(-1)-3(1) \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 1 \end{bmatrix}}$$

Ex.12 $\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}$

$2\times 2 \quad 3\times 2$

undefined

Inverses of 2x2 Matrices

- Identity matrix- consist of ones on the main diagonal and zeros elsewhere.
- The identity matrix must be square I

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Determinants- every square matrix can be associated with one real number.
- A matrix has an inverse if the determinate is not equal to zero.
- To find the determinate of a 2x2 matrix, find the difference of the products of the two diagonals.

$$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

Ex.1 Find the determinate of matrix A.

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} = 2(2) - 1(-3) = 7$$

Finding inverses of 2x2 matrices

- **If a matrix has an inverse, it's called invertable.**
- $AA^{-1} = I = A^{-1}A$

$$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \quad A^{-1} = \frac{1}{ad-cb} \begin{vmatrix} d & -b \\ -c & a \end{vmatrix}$$

Ex.2 Find the inverse for matrix A and B.

$$A = \begin{vmatrix} 3 & -1 \\ -2 & 2 \end{vmatrix} \qquad B = \begin{vmatrix} 3 & -1 \\ -6 & 2 \end{vmatrix}$$

$$\det(A) = 3(2) - (-2)(-1)$$
$$= 4$$

$$A^{-1} = \frac{1}{4}\begin{vmatrix} 2 & 1 \\ 2 & \frac{3}{1} \end{vmatrix}$$

$$A^{-1} = \begin{vmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{vmatrix}$$

$$\det(B) = 6 - 6$$
$$= 0$$

not invertible

Ex.3 Finding the determinate of a 3x3 matrix

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\det(A) = 0\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix}$$

$$= 0 - 2(3-8) + 1(0+4)$$

$$\det(A) \quad -2(-5) +$$

$\det(A) = -2(-5) +$

$= 10 + 4$

$\boxed{\det(A) = 14}$

<u>Solving Systems of Linear equations</u>

- **Systems of equations have exactly one solution, infinitely many solutions, or no solution.**
- **If A is an invertible matrix, the stream of Linear equations represented by AX = B has a unique solution.** $X = A^{-1}B$

Ex.1 Solve the system of equations using an inverse.

$$\begin{cases} -3x - 4y = -5 \\ 4x + 3y = 9 \end{cases}$$

$$\begin{bmatrix} -3 & -4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 9 \end{bmatrix}$$

$\det(A) = -3(3) - (-4)(4)$

$= -9 + 16$

$= 7$

$$\frac{1}{7}\begin{bmatrix} 3 & 4 \\ -4 & -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{7} & \frac{4}{7} \\ -\frac{4}{7} & -\frac{3}{7} \end{bmatrix} \begin{bmatrix} -5 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \left(\frac{3}{7}\right)(-5) + \left(\frac{4}{7}\right)(9) \\ -\frac{4}{7}(-5) + \left(-\frac{3}{7}\right)(9) \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \qquad \boxed{\begin{array}{l} x = 3 \\ y = -1 \end{array}}$$

Ex.2

$$\begin{cases} 2x + 3y + z = -1 \\ 3x + 3y + z = 1 \\ 2x + 4y + z = -2 \end{cases}$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}}$$

Area of a Triangle

- **We can use the determinate to find the area of a triangle.**
- **Form a 3x3 determinate where the first column are the x's for all the points, the second column are the y's for all the points, and the last column is all ones.**

$$\text{Area Triangle} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

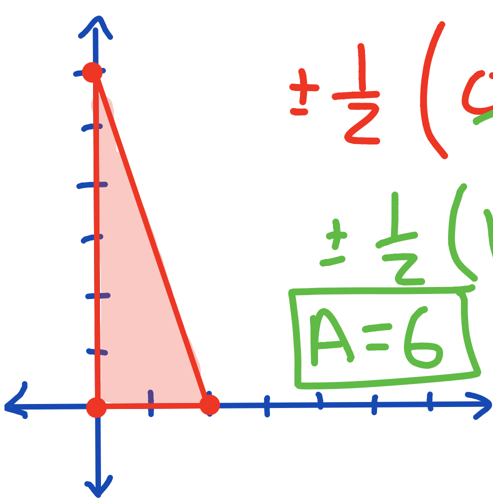
Ex.1 Find the area of a triangle with vertices (0,0), (0,6), and (2,0).

$$A = \pm\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 0 & 6 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

$$\pm\frac{1}{2}\left(\cancel{0\begin{vmatrix} 6 & 1 \\ 0 & 1 \end{vmatrix}} \cancel{- 0\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix}} + 1\begin{vmatrix} 0 & 6 \\ 2 & 0 \end{vmatrix}\right)$$

$$\pm\frac{1}{2}(12)$$

$$\boxed{A = 6}$$



Transformations Using Matrices

- **Polygons can be represented in matrix form by placing all of the vertices in one matrix. The x's are placed in row one and the y's in row 2.**

Ex.1 Write a matrix for a square that has the following coordinates (1,1), (-1,1), (-1,-1), and (1,-1).

$$\begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

Translations

- **If we want to translate the figure left 3 and up 2, we add 3 to each x and 2 to each y.**

$$\begin{bmatrix} x_1+3 & x_2+3 & x_3+3 & x_4+3 \\ y_1+2 & y_2+2 & y_3+2 & y_4+2 \end{bmatrix}$$

Dilations

- To dilate a figure, multiply each x and y coordinate by a scale factor.

$$3\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{bmatrix}$$

Reflections

- To reflect an image, multiply the vertex matrix by the reflection matrix.
- Reflection across the x-axis

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- Reflection across the y axis.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Reflection across y=x.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Ex.2 Reflect the segment across the x axis

$$A = \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 3 \\ -2 & 2 \end{bmatrix}$$

Rotations

- To rotate a figure, multiply by the rotation matrix.
- Counter clockwise 90

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

- **Counter clockwise 180**

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

- **Counter clockwise 270**

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$