Trigonometric Identities

- A statement of equality between two expressions are defined is called an identity.

Reciprocal Identities

$\sin\theta = \frac{1}{\csc\theta}$ $\cos\theta = \frac{1}{\sec\theta}$ $\tan\theta = \frac{1}{\cot\theta}$

$\csc\theta = \frac{1}{\sin\theta}$ $\sec\theta = \frac{1}{\cos\theta}$ $\cot\theta = \frac{1}{\tan\theta}$

Quotient Identities

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

*Try putting everything in terms of sine and cosine first

Ex.1 Prove that sin(x)csc(x) = 1

$$\frac{\sin x}{1} \cdot \left(\frac{1}{\sin x}\right) = 1$$

$$\frac{\sin x}{\sin x} = 1$$

$$1 = 1$$

Ex.2 Simplify sin(x) + sin(x)cot^2(x)

$$\sin x\left(1 + \cot^2 x\right)$$

$$\sin x\left(\csc^2 x\right)$$

$$\frac{\sin x}{1}\left(\frac{1}{\sin^2 x}\right)$$

$$\frac{\cancel{\sin x}}{\sin^{\cancel{2}} x}$$

$$\frac{1}{\sin x} = \boxed{\csc x}$$

Ex.3 verify that $\sec^2(x) - \tan(x)\cot(x) = \tan^2(x)$

$$\sec^2 x - \frac{\cancel{\sin x}}{\cancel{\cos x}} \cdot \frac{\cancel{\cos x}}{\cancel{\sin x}} = \tan^2 x$$

$$\sec^2 x - 1 = \tan^2 x$$

$$\tan^2 x = \tan^2 x \checkmark$$

Ex.4 verify that...

$$\frac{7\sin\theta + 5\cos\theta}{\sin\theta\cos\theta} = 7\sec\theta + 5\csc\theta$$

$$\frac{7\cancel{\sin\theta}}{\cancel{\sin\theta}\cos\theta} + \frac{5\cancel{\cos\theta}}{\sin\theta\cancel{\cos\theta}} = 7\sec\theta + 5\csc\theta$$

$$\frac{7}{\cos\theta} + \frac{5}{\sin\theta} = 7\sec\theta + 5\csc\theta$$

$$7\sec\theta + 5\csc\theta = 7\sec\theta + 5\csc\theta \checkmark$$

Sum and Difference Identities

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

Ex.1 Use the sum and difference Identities to find the exact value:

a) sin(15) $= \sin(45-30)$

$$\sin 45 \cdot \cos 30 - \cos 45 \cdot \sin 30$$

$$\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$

$$\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \boxed{\frac{\sqrt{6}-\sqrt{2}}{4}}$$

b) $\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

$$= \cos\frac{\pi}{4} \cdot \cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6}$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$

$$= \boxed{\frac{\sqrt{6}+\sqrt{2}}{4}}$$

c) $\tan(15) = \tan(45-30)$

$$= \frac{\tan 45 - \tan 30}{1 + \tan 45 \cdot \tan 30}$$

$$= \frac{1 - \frac{\sqrt{3}}{3}}{1 + 1 \cdot \frac{\sqrt{3}}{3}}$$

$$= \boxed{-\sqrt{3}+2}$$

$$\frac{1}{2} \div \frac{\sqrt{3}}{2}$$

$$\frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\sqrt{3}}$$

$$\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$

$$\frac{\sqrt{3}}{3}$$

Ex.2 Rewrite the expression using sin, cos, or tan: sin(340)cos(50) - cos(340)sin(50)

$$\sin(340+50)$$

$$\boxed{\sin(390)}$$

Ex.3 Find the exact value of the trig function given:

Find cos(u + v)

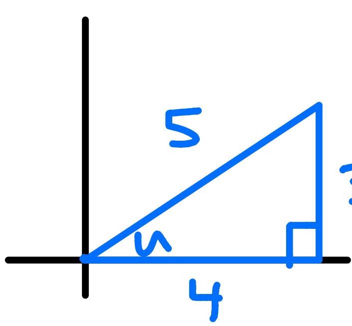
$$\sin u = \frac{5}{3} \quad 0 < u < \frac{\pi}{2}$$

$$\cos v = \frac{3}{5} \quad \frac{3\pi}{2} < v < 2\pi$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

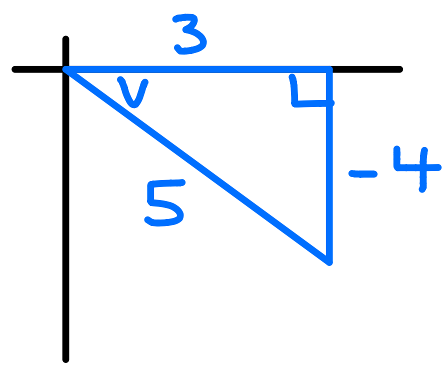
$$\cos u \cdot \cos v - \sin u \cdot \sin v$$

$$\left(\frac{4}{5}\right)\cdot\left(\frac{3}{5}\right) - \left(\frac{3}{5}\right)\left(\frac{-4}{5}\right)$$



$$\frac{12}{25} + \frac{12}{25}$$

$$\boxed{\frac{24}{25}}$$



$a^2+b^2=c^2$
$3^2+b^2=5^2$
$b=4$

$$\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta$$

Ex.4 verify

$$\sin\left(x+\frac{\pi}{3}\right)+\sin\left(x-\frac{\pi}{3}\right)=\sin x$$

$$\sin x\cdot\cos\frac{\pi}{3}+\cos x\sin\frac{\pi}{3}+\sin x\cdot\cos\frac{\pi}{3}-\cos x\sin\frac{\pi}{3}$$

$$\sin x\left(\frac{1}{2}\right)+\cos x\left(\frac{\sqrt{3}}{2}\right)+\sin x\left(\frac{1}{2}\right)-\cos x\left(\frac{\sqrt{3}}{2}\right)$$

$$\frac{1}{2}\sin x+\cancel{\frac{\sqrt{3}}{2}\cos x}+\frac{1}{2}\sin x-\cancel{\frac{\sqrt{3}}{2}\cos x}=\sin x$$

$\sin x=\sin x$ ✓

Double Angle and Half Angle Identities

- It is useful to have Identities to find the value of a function of twice and Angle or half an angle.

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

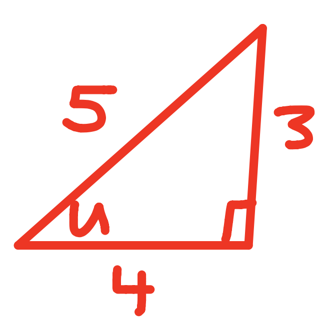
Ex.1 Find the exact value of sin(2u) and tan(2u).

$$\sin u = \frac{3}{5} \qquad 0 < u < \frac{\pi}{2}$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\sin 2u = \frac{2}{1}\left(\frac{3}{5}\right)\left(\frac{4}{5}\right)$$

$$\sin 2u = \boxed{\frac{24}{25}}$$

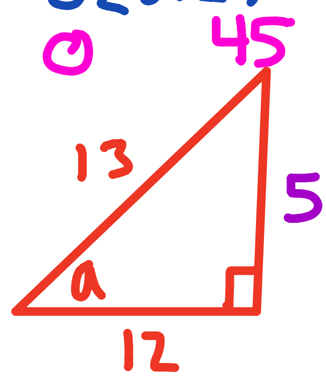


$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\tan 2u = \frac{2\left(\frac{3}{4}\right)}{1-\left(\frac{3}{4}\right)^2} = \frac{\frac{6}{4}}{1-\frac{9}{16}} = \frac{\frac{6}{4}}{\frac{7}{16}} = \boxed{\frac{24}{7}}$$

Ex.2 Find the value of $\sin\left(\frac{a}{2}\right)$

$$\cos a = \frac{12}{13} \quad \text{where } 0 < a < 90^\circ$$



$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\sin\frac{a}{2} = \pm\boxed{\sqrt{\frac{1-\left(\frac{12}{13}\right)}{2}}}$$

$$\sqrt{\frac{1}{13} \div \frac{2}{1}}$$

$$\sqrt{\frac{1}{13} \cdot \frac{1}{2}}$$

$$\pm\sqrt{\frac{1}{26}}$$

$$\frac{1}{\sqrt{26}} \cdot \frac{\sqrt{26}}{\sqrt{26}} = \boxed{\frac{\sqrt{26}}{26}}$$

$$a^2 + b^2 = c^2$$

$$12^2 + b^2 = 13^2$$

$$b^2 = 13^2 - 12^2$$

$$\sqrt{b^2} = \sqrt{25}$$

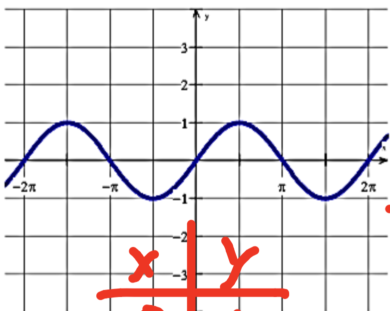
$$\boxed{b = 5}$$

Inverse Trig Functions Notes

Arcsine: (Sine Inverse)

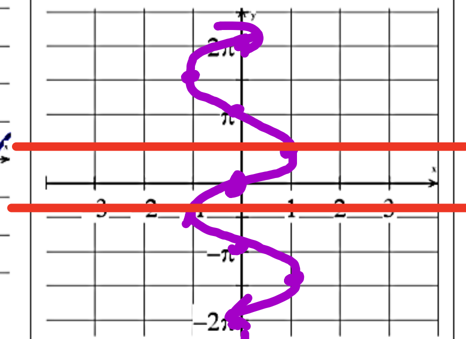
$\sin\theta = c$ → $\sin^{-1} \quad =$ **OR** $\arcsin \quad =$

Graph of $y = \sin\theta$



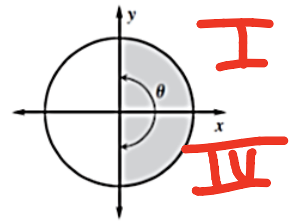
x	y
0	0
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	-1
2π	0

Graph of $\theta = \sin^{-1} c$



x	y
0	0
1	$\frac{\pi}{2}$
0	π
-1	$\frac{3\pi}{2}$
0	2π

Values of θ used for arcsine:



Domain of arcsine:

$[-1, 1]$

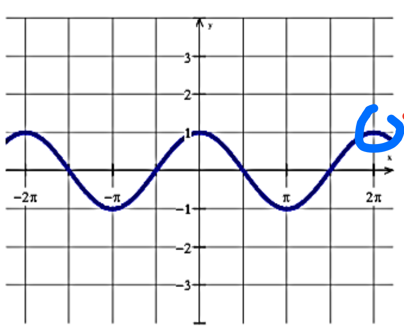
Range of arcsine:

$[-\frac{\pi}{2}, \frac{\pi}{2}]$

Arccosine: (Cosine Inverse)

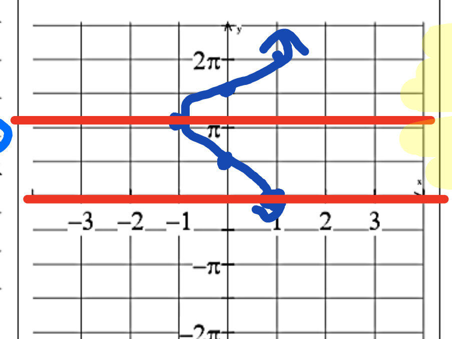
$\cos\theta = c$ → $\cos^{-1} c = \theta$ **OR** $\arccos c = \theta$

Graph of $y = \cos\theta$



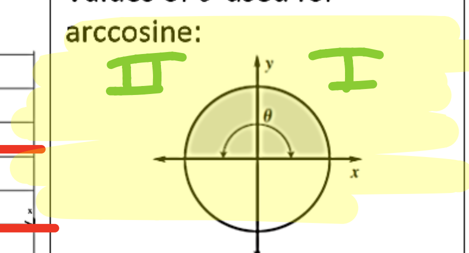
x	y
0	1
$\frac{\pi}{2}$	0
π	-1
$\frac{3\pi}{2}$	0
2π	1

Graph of $\theta = \cos^{-1} c$



x	y
1	0
0	$\frac{\pi}{2}$
-1	π
0	$\frac{3\pi}{2}$
1	2π

Values of θ used for arccosine:



Domain of arccosine:

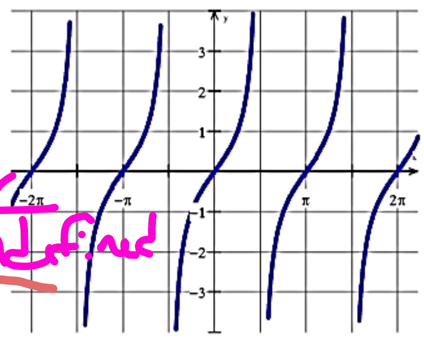
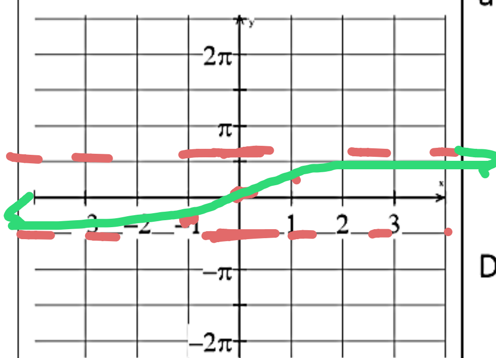
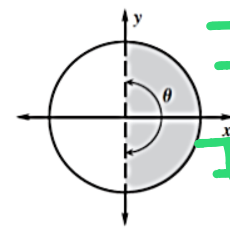
$[-1, 1]$

Range of arccosine:

$[0, \pi]$

Arctangent: (Tangent Inverse)

$\tan\theta = c \longrightarrow \tan^{-1} c = \theta$ **OR** $\arctan c = \theta$

Graph of $y = \tan\theta$	Graph of $\theta = \tan^{-1} c$	Values of θ used for arctangent:
		 I IV
		Domain of arctangent: $(-\infty, \infty)$
		Range of arctangent: $[-\frac{\pi}{2}, \frac{\pi}{2}]$

x	y
$-\frac{\pi}{2}$	undefind
$-\frac{\pi}{4}$	-1
0	0
$\frac{\pi}{4}$	1
$\frac{\pi}{2}$	undef:nd

Find each value.

1. $\arcsin(-\frac{\sqrt{2}}{2})$ that angle whose sin is $-\frac{\sqrt{2}}{2}$

2. $\sin^{-1} 0$

3. $\tan^{-1}\frac{\sqrt{3}}{3}$

4. $\sin^{-1} 2$

5. $\sin^{-1}(\cos(\frac{\pi}{2}))$

6. $cos(\tan^{-1}\sqrt{3})$

7. $\cot^{-1}(2)$

Solving Trigonometric Equations

- Most trig equations have more than one solution. The periodic nature will result in an infinite number of solutions.
- Many trig expressions will have two values for one period.

Solve each equation for $0 \leq x < 2\pi$

Ex.1 solve $2\sin(x) + \cancel{1} = 0$

$$\cancel{-1} \quad -1$$

$$\frac{\cancel{2}\sin(x)}{\cancel{2}} = \frac{-1}{2}$$

$$\sin(x) = -\frac{1}{2}$$

$$\boxed{\begin{array}{l} x = 210^\circ \\ x = 330^\circ \end{array}}$$

Ex.2 solve $\sin(x)\cos(x) - \frac{1}{2}\cos(x) = 0$

$$\cos x \left(\sin x - \frac{1}{2}\right) = 0$$

$$\cos x = 0$$

$$\boxed{x = \frac{\pi}{2}, \frac{3\pi}{2}}$$

$$\sin x - \cancel{\frac{1}{2}} = 0$$

$$\cancel{+\frac{1}{2}} \quad +\frac{1}{2}$$

$$\sin x = \frac{1}{2}$$

$$\boxed{x = \frac{5\pi}{6} \quad x = \frac{\pi}{6}}$$

Ex.3 $\sin(2x) - \cancel{1} = 0$

$+\cancel{1} \quad +1$

$\sin(2x) = 1$

$\frac{\cancel{2}x}{\cancel{2}} = \frac{\frac{\pi}{2}}{2}$

$\boxed{x = \frac{\pi}{4}}$

Ex.4 $\sin(x) + \cos(x) = 0$

$\boxed{x = \frac{3\pi}{4} \qquad x = \frac{7\pi}{4}}$

Ex.5 $\sqrt{2}\cos(x) + \cancel{1} = 0$

$-\cancel{1} \quad -1$

$\frac{\cancel{\sqrt{2}}\cos(x)}{\cancel{\sqrt{2}}} = \frac{-1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{-\sqrt{2}}{2}$

$\cos(x) = -\frac{\sqrt{2}}{2}$

$\boxed{x = \frac{3\pi}{4} \qquad x = \frac{5\pi}{4}}$

Ex.6 $\sin(x)\tan(x) - \sin(x) = 0$

$\sin x\,(\tan x - 1) = 0$

$\sin x = 0 \qquad \tan x - \cancel{1} = 0$

$+\cancel{1} \quad +1$

$\boxed{x = 0 \quad x = \pi} \qquad \tan x = 1 \qquad \boxed{x = \frac{\pi}{4}}$

$$\boxed{x = \frac{5\pi}{4}}$$

Ex.7 $\cos^2(x) = \cos(\cancel{x})$

$-\cos x \quad -\cancel{\cos x}$

$\cos^2 x - \cos x = 0$

$\cos x\,(\cos x - 1) = 0$

$\cos x = 0 \qquad \cos x - \cancel{1} = 0$

$\qquad\qquad\qquad \cancel{+1} \quad +1$

$\boxed{x = \frac{\pi}{2} \quad x = \frac{3\pi}{2}} \qquad \cos x = 1 \qquad \boxed{x = 0}$

Ex.8 $\sec^2(x) - \cancel{2} = 0$

$\cancel{+2} \quad +2$

$\sqrt{\sec^2 x} = \sqrt{2}$

$\sec x = \frac{\sqrt{2}}{1}$

$\cos x = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$

$$\boxed{\begin{array}{l} x = \frac{\pi}{4} \\ x = \frac{7\pi}{4} \\ x = \frac{5\pi}{4} \\ x = \frac{3\pi}{4} \end{array}}$$

Ex.9 $\cos(x)\tan(x) = \frac{1}{2}$

$\frac{\cancel{\cos x}}{1} \cdot \frac{\sin x}{\cancel{\cos x}} = \frac{1}{2}$

$\sin x = \frac{1}{2}$

$\boxed{x = \frac{\pi}{6} \qquad x = \frac{5\pi}{6}}$

Ex.10 $2\cos^2(x) - 5\cos(x) + 2 = 0$ $[0, 360°]$

$2 \cdot 2 = 4$

$1 \cdot 4$

$2 \cdot 2$

$2x^2 - 5x + 2 = 0$

$(2x - 1)(2x - 4) = 0$

$(2\cos x - 1)(2\cos x - 4) = 0$

$2\cos x - \cancel{1} = 0$

$\cancel{+1} \quad +1$

$\frac{\cancel{2}\cos x}{\cancel{2}} = \frac{1}{2}$

$\cos x = \frac{1}{2}$

$\boxed{x = 60° \quad x = 300°}$

$2\cos x - \cancel{4} = 0$

$\cancel{+4} \quad +4$

$\frac{\cancel{2}\cos x}{\cancel{2}} = \frac{4}{2}$

$\cos x = 2$

Solving Trigonometric Equations with Caluculator

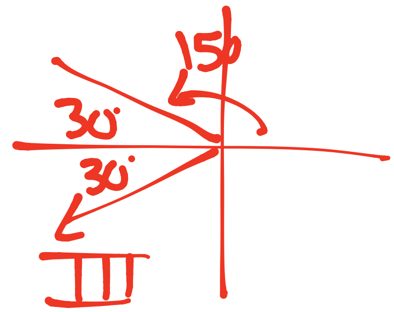
Ex.1 $2\cos(x) + \sqrt{3} = 0$ $\quad [0^\circ, 360^\circ]$

$-\sqrt{3} \quad -\sqrt{3}$

$$\frac{\cancel{2}\cos(x)}{\cancel{2}} = \frac{-\sqrt{3}}{2}$$

$$\cancel{\cos^{-1}}\ \cancel{\cos} x = \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) \quad \boxed{210}$$

$$\boxed{x = 150}$$



Ex.2 $\sec(x+81) = 2$ $\quad [0^\circ, 360^\circ]$

$$\frac{1}{\cos}(x+81) = \frac{2}{1}$$

$$\cos(x+81) = \frac{1}{2}$$

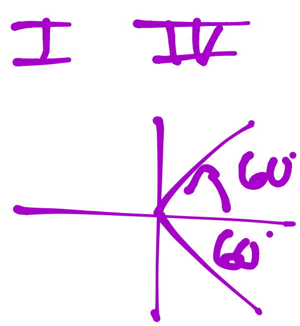
$$\cancel{\cos^{-1}}\ \cancel{\cos}(x+81) = \cos^{-1}\left(\frac{1}{2}\right)$$

$x+81 = 60 \quad x+81 = 300 = 60, 300^\circ$

$-21 \quad \boxed{219}$

$+360$

$\boxed{339}$



Ex.3 $\frac{\cancel{4}\cos^2(x)}{\cancel{4}} = \frac{3}{4}$ $\quad [90^\circ, 180^\circ]$

$$\sqrt{\cos^2 x} = \sqrt{\frac{3}{4}}$$

$$\cos x = \cancel{\pm}\frac{\sqrt{3}}{2}$$

$$\cancel{\cos^{-1}\cos} x = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

$$\boxed{x = 150^\circ}$$

Ex.4 tan(x)sec(x) = tan(~~x~~) $[0, 360^\circ]$

$$-\tan x \qquad -\tan x$$

$$\tan x \sec x - \tan x = 0$$

$$(\tan x)(\sec x - 1) = 0$$

$$\tan x = 0$$

$$\tan^{-1} 0 = x$$

$$\boxed{x = 0^\circ, 360^\circ}$$

$$\sec x - \cancel{1} = 0$$

$$\cancel{+1} \quad +1$$

$$\sec x = 1$$

$$\frac{1}{\cos x} = \frac{1}{1}$$

$$\cos x = 1$$

$$\cos^{-1} 1 = x$$

$$\boxed{x = 0^\circ, 360^\circ}$$

Ex.5 $2\cos^2(x) - 5\cos(x) + 2 = 0$ $[0°, 360°]$

$2x^2 - 5x + 2 = 0$ $ax^2 + bx + c = 0$

$a \cdot c$

4

$-1 \cdot -4$

$2 \cdot 2$

$(2x - 1)\left(\frac{2x}{2} - \frac{4}{2}\right) = 0$

$(2x - 1)(x - 2) = 0$

$(2\cos x - 1)(\cos x - 2) = 0$

$2\cos x - 1 = 0$

$\cos x = \frac{1}{2}$

$\boxed{x = 60° \quad 300°}$ I IV

$\cos x - 2 = 0$

$\cos x = 2$

No Solution